ON THE ELASTO-PLASTIC WAVE PROPAGATION IN A HALF-SPACE

(O RASPROSRANENII UPRUGO-PLASTICHESKIKH VOLN V POLUPROSTRANSTVE)

PMM Vol.23, No.3, 1959, pp.419-424

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(Received 17 February 1959)

1. Formulation of the problem. Let the pressure $p(x, y, t)$ be acting on the boundary $z = 0$ of a half-space xyz. As is known, two time-dependent regions can be distinguished in an elastic half-space. In one region there will be only longitudinal waves, while in the other region longitudinal as well as transverse waves will exist.

We note that the front of the transverse waves will be a wave of a weak discontinuity. For the case of elasto-plastic wave propagation the qualitative picture of motions will be the same.

An approximate method of solution is given below for the problem of propagation of waves of weak discontinuity in a half-space, the boundary of which is subjected to the action of a normal pressure.

First, we note that the essential factor will be the displacement along the z-axis in direction of the pressure on the boundary.

For $u = v = 0$, the dynamic equations of the theory of elasticity are

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial z^2} + b^2 \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\right) \tag{1.1}$$

From the form of this equation it is seen that the forward front of the disturbed region is different from the same obtained from the exact solution. We propose to improve equation (1.1) so that the form of the forward front would coincide with the actual one.

Clearly in this case (1.1) will have the form

$$\frac{\partial^2 w}{\partial t^2} = a^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{1.2}$$

It has to be emphasized that our formulation of the problem is different from the acoustical version of the dynamic problem of the theory of elasticity known in the literature.

Although (1.2) represents the wave equation, the boundary conditions, however, are different from the boundary conditions for the displacement potential, ϕ. Besides, the stresses are expressed by w in a different way than by ϕ. Moreover, solutions of (1.2) for a given value of the normal derivative $\partial w/\partial z$ has a simpler form than for ϕ. Even the qualitative solutions of our formulation of the problem will substantially differ from the acoustic formulation.

Indeed, the acoustic theory corresponds to the motion of an elastic fluid. As a consequence of this, if on the portion of the boundary where the pressure is applied there appears a depression, in opposition thereto there will appear a bulge in the free portion. This is not observable in the elastic half-space. Many properties as well as the solution of (1.2) are close to reality.

We note finally that the solution of the static problem justifies the assumption $u = v = 0$.

2. Solution of an approximate problem of wave propagation in an elastic half-space. Following Rayleigh, we shall seek solutions of (1.2) in the following form:

$$w = \iiint\limits_{\omega} \frac{C(\xi, \eta, \tau)\, d\xi \eta\, d\tau}{\sqrt{(\xi - x)^2 + (\eta - y)^2 + z^2}} \tag{2.1}$$

The region of integration, ω, is a part of a half-space bounded by the surface of a hyperboloid

$$\tau = t - \frac{1}{a}\sqrt{(\xi - x)^2 + (\eta - y)^2 + z^2} \tag{2.2}$$

and the plane $r = 0$. It is easy to demonstrate the following equality

$$\left[\frac{\partial w}{\partial z}\right]_{z=0} = -2\pi \int_0^t C(x, y, \tau)\, d\tau \tag{2.3}$$

Note that if $\partial w/\partial z$ = const. for $z = 0$, then from (2.3) we obtain $C(x, y, t) \equiv 0$, assuming that $C(x, y, t)$ is a continuous function of t.

This difficulty is overcome if the condition of continuity of C is abandoned, and if we put

$$C(x, y, t) = A, \delta(t) + B(x, y, t) \tag{2.4}$$

where $\delta(t)$ is Dirac's function; $B(x, y, t)$, is a continuous function of t. Then (2.4) yields

$$\left.\frac{\partial w}{\partial z}\right|_{z=0} = -A - 2\pi\int_0^t B(x, y, \tau)\,d\tau \qquad (2.5)$$

From (2.5) we find

$$B(x, y, t) = -\frac{1}{2\pi}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right)_{z=0} \qquad (2.6)$$

Substituting (2.4) and (2.6) in (2.1) we get

$$w(x, y, z, t) = \iiint_\omega \frac{A, \delta(\tau)\,d\xi\,d\eta\,d\tau}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}} - \frac{1}{2\pi}\iiint_\omega \frac{w_{z\tau}(x, y, 0\tau)\,d\tau\,d\xi\,d\eta}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}} \qquad (2.7)$$

Consider now a case of wave propagation due to a suddenly applied axi-symmetric load, which remains constant after application.

The solution will have the form:

$$w(x, y, z, t) = A\iiint_\omega \frac{\delta(\tau)\,d\tau\,d\xi\,d\eta}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}} \qquad (2.8)$$

Using (2.3) and (2.4) we find

$$A = -\left[\frac{\partial w}{\partial z}\right]_{z=0}\frac{1}{2\pi} = -\frac{\varepsilon}{2\pi} \qquad (2.9)$$

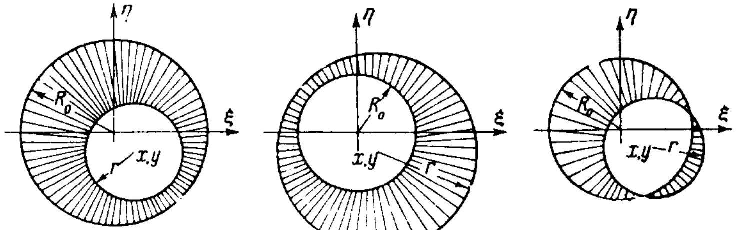
The region of integration in (2.7) is a common volume of a cylinder of radius R_0 (radius of the load) and of the hyperboloid (2.2).

During the process of integration with respect to r, the variables ξ and η are separable. Consequently, in the $\xi\eta$-region the three cases of the relative position of the circles

$$(\xi-x)^2+(\eta-y)^2 = a^2t - z^2 = r^2, \qquad \xi^2+\eta^2 = R_0{}^2$$

are possible.

The integration takes place along a common part of the circles with the radii R_0 and r.



Case 1 *Case 2* *Case 3*

Fig. 1

Case 1. Introducing polar coordinates we obtain

$$w = -\frac{\varepsilon}{2\pi}\int_1^{\rho}\frac{2\pi r\,dr}{\sqrt{r^2+z^2}} = -\varepsilon(at - z)$$

$$(\rho = \sqrt{a^2t^2 - z^2}) \tag{2.10}$$

We now explain where this solution is valid. For the case 1 it is seen from Fig. 1 that

$$\sqrt{x^2+y^2} + \sqrt{a^2t^2 - z^2} \leqslant R_0$$

We have $x^2 + y^2 = r^2$, and consequently

$$at \leqslant \sqrt{(R_0 - r)^2 + z^2} \tag{2.11}$$

Since at is the distance from the forward wave front, then (2.11) means that the solution (2.9) is valid in the shaded areas of Fig. 2. This is in complete agreement with the physical picture of the motion. Thus it is shown that the plane part of the forward wave front carries a pressure which is applied at the initial moment.

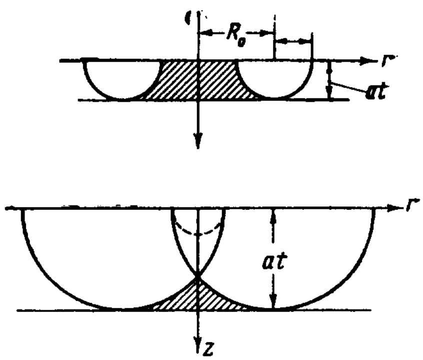


Fig. 2.

Case 2. Let us split the region of integration into two parts, as shown in Fig. 3. Putting $x^2 + y^2 = r^2$ and $a^2t - y^2 = \rho^2$, θ_1 can then be found from

$$\sin\theta_1 = \sqrt{1 - \left[\frac{R_0^2 - \rho^2 - r^2}{2\rho r}\right]^2} \tag{2.12}$$

The equation of the polar radius in the region II will be

$$R = \sqrt{R_0^2 - r^2 + r^2\cos^2\theta} - r\cos\theta \tag{2.13}$$

By means of the formulas (2.8) and (2.9) we obtain

$$w(x, y, z, t) = -\frac{\varepsilon}{2\pi} \iint\limits_{II+I} \frac{d\xi\, d\eta}{\sqrt{(\xi - x)^2 + (\eta - y)^2 + z^2}}$$

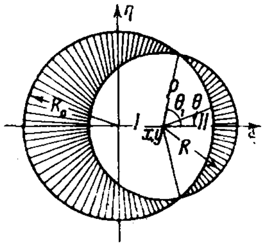


Fig. 3

Using polar coordinates we have

$$w = -\frac{\varepsilon}{2\pi} \int\limits_{\theta_1}^{2\pi-\theta_1} d\theta \int\limits_0^{\rho} \frac{r\,dr}{\sqrt{r^2 + z^2}} - \frac{\varepsilon}{2\pi} \int\limits_{-\theta_1}^{\theta_1} d\theta \int\limits_0^{R} \frac{r\,dr}{\sqrt{r^2 + z^2}} =$$

$$= -\frac{\varepsilon}{2\pi} [\sqrt{\rho^2 + z^2} - z](2\pi - 2\theta_1) - \frac{\varepsilon}{2\pi} \int\limits_{-\theta_1}^{\theta_1} [\sqrt{R^2 + z^2} - z]\, d\theta$$

Since $\rho^2 = a^2 t^2 - z^2$, then

$$w = -\frac{\varepsilon}{2\pi}(at - z)(2\pi - 2\theta_1) - \frac{\varepsilon}{2\pi} \int\limits_{-\theta_1}^{+\theta_1} [\sqrt{R^2(\theta) + z^2} - z]\, d\theta \tag{2.14}$$

Case 3. Note that $r - \rho > R_0$, $r > R_0$, and furthermore (Fig. 4)

$$R_0 = r^2 + \rho^2 - 2r\rho\cos\theta_2, \quad \cos\theta_2 = \frac{r^2 + \rho^2 - R_0^2}{2\rho r}$$

$$r^2 + R^2 - 2rR\cos\theta - R_0^2 = 0$$

$$R = r\cos\theta - \sqrt{r^2\cos^2\theta + R_0^2 - r^2}$$

We have therefore

$$w = -\frac{\varepsilon}{2\pi} \int\limits_{-\theta_2}^{\theta_2} d\theta \int\limits_R^{\rho} \frac{r\,dr}{\sqrt{r^2 + z^2}} \int\limits_{-\theta_2}^{\theta_2} [\sqrt{\rho^2 + z^2} - \sqrt{R^2 + z^2}]\, d\theta$$

or

$$w = -\frac{\varepsilon a t}{2\pi} 2\theta_2 + \frac{\varepsilon}{2\pi} \int\limits_{-\theta_2}^{\theta_2} \sqrt{R^2 + z^2}\, d\theta \tag{2.15}$$

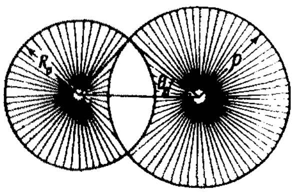


Fig. 4.

Let us consider the one-dimensional problem. Here $R_0 = \infty$ and we have only the first case. For simplicity assume that $\partial w/\partial r = 0$ for $z = 0$, $t = 0$. Then from (2.7) we obtain

$$w = -\frac{1}{2\pi}\int_0^{\tau_1} \varepsilon'_\tau(\tau)\,d\tau \int_0^{2\pi} d\theta \int_0^{\rho_1} \frac{r\,dr}{\sqrt{r^2+z^2}} = -\int_0^{\tau_1} \varepsilon'_\tau(\tau)\,[a(t-\tau)-z]\,d\tau$$

$$\left(\tau_1 = t - \frac{z}{a},\ \rho_1 = \sqrt{a^2(t-\tau)^2 - z^2}\right)$$

From here

$$w = -\int_0^{\tau_1} a\varepsilon(\tau)\,d\tau, \qquad w_t = -a\varepsilon\left(t-\frac{z}{a}\right) = f\left(t-\frac{z}{a}\right) \tag{2.16}$$

where $f(t)$ is the value of the velocity on the boundary.

3. An approximate solution of the problem of elasto-plastic wave propagation in a half-space. The same reasoning as for the case of elastic wave propagation will lead to the conclusion that for the cases of the elasto-plastic waves a substantial role will be played by the longitudinal displacements, i.e. displacements in the direction of the given load on the boundary.

If we neglect the transverse waves, then in this case it can be assumed that

$$\frac{\partial^2 w}{\partial t^2} = a^2(e)\left[\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\right] \tag{3.1}$$

where e is the intensity of the deformation.

For Prandtl's model (3.1) is obviously linear, but only with discontinuous coefficients.

First we note that in this case e has the form

$$e=\sqrt{\frac{2}{3}}\sqrt{\left(\frac{\partial w}{\partial z}\right)^2+\frac{3}{2}\left(\frac{\partial w}{\partial r}\right)^2} \tag{3.2}$$

From this it can be seen that in the region where the influence of the boundary has propagated, the stress intensity is increasing, at least near the boundary.

Assume that the pressure of $(\partial w/\partial r)_{z=q} = \epsilon$ is so large that $|\partial w/\partial r| \gg \epsilon_s$. In this case we will have plastic deformation in the whole region where the boundary exhibits its influence. The picture of elasto-plastic wave propagation, accepting Prandtl's model, is shown in Fig. 5.

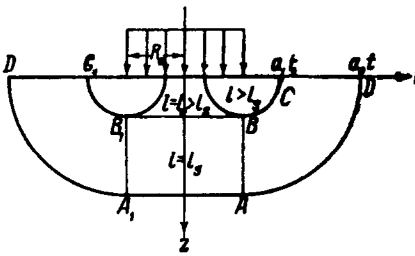


Fig. 5.

The only region of elastic deformation is the region $ABCD$ (Fig. 5), while the surface C_1B_1BC is a surface of discontinuity of deformations.

It is seen from Fig. 5 that the process is active everywhere. Notice that only the pressure penetrates deep into the body, while large deformations are concentrated in plastic regions. This is so because for a large increase of ϵ the stresses are changed insignificantly.

We now consider the process to be rigid-plastic. In this case the forward wave front will be the fron of plastic waves. The solution will be the same as in case of propagation of elastic deformations, except that instead of the elastic velocity of deformations, a, we substitute the velocity of plastic deformations, a_1.

Consider now the case of elasto-plastic mechanism of motion. We represent a solution by two functions:

$$w_1=\iiint\limits_{\omega}\frac{C(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}} \tag{3.3}$$

$$w_2=\iint\limits_{\sigma}\frac{A(\xi,\eta)\,d\xi\,d\eta}{\sqrt{(\xi-x)^2+(\eta-y)^2+z}}+\iiint\limits_{\Omega}\frac{B(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau}{\sqrt{(\xi-x)^2+(\eta-y)^2z^2}} \tag{3.4}$$

where ω is an intersection of a cylinder with the base equal to the area of the load with the hyperboloid (2.1), and a_1 replaces a.

The region σ is a common part of the circles

$$\xi^2 = \eta^2 = R_0^2, \qquad (\xi - x)^2 + (\eta - y)^2 + z^2 = a^2 t^2$$

The region Ω is the intersection of the above-mentioned cylinder and the hyperboloid (2.1).

The functions A, B and C are found from

$$\left[\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial z}\right]_{z=0} = -2\pi A_1(x, y) - 2\pi \int_0^t C(x, y, \tau)\, d\tau - 2\pi \int_0^t B(x, y, \tau)\, d\tau$$

From this equation it is seen that for the initial deformation along the line $C_1 B_1 B C$ we have

$$e_s = \frac{2}{3} \sqrt{\left(\frac{\partial w_2}{\partial z}\right)^2 + \frac{3}{2}\left(\frac{\partial w_2}{\partial}\right)^2}$$

Thus, in the region of plastic deformations a solution is

$$w = w_1 + w_2 \tag{3.6}$$

and in the region of elastic deformations

$$w = w_2 \tag{3.7}$$

4. Analysis of the state of stress near the boundary of the leading region. The displacement in the plastic half-space near the boundary of the loading region is determined by (2.14), where a_1 replaces a. Rewrite this relationship as

$$w = -\frac{\varepsilon a_1 t}{\pi} \theta_2 + \frac{\varepsilon}{2\pi} \int_{-\theta_2}^{\theta_2} \sqrt{R^2 + r^2}\, d\theta \tag{4.1}$$

where

$$\cos \theta_2 = \frac{r^2 + \rho^2 - R_0}{2\rho r}, \qquad \rho^2 = a^2 t^2 - z^2 \tag{4.2}$$

From (4.2), differentiating with respect to z and r, we obtain

$$-\sin \theta_2 \frac{\partial \theta_2}{\partial z} = -\frac{z}{\rho r} + \frac{r^2 - \rho^2 - R_0}{2r\rho^2} \frac{z}{\rho}, \qquad -\sin \theta_2 \frac{\partial \theta_2}{\partial r} = \frac{1}{\rho} - \frac{r^2 + \rho^2 - R_0^2}{2\rho r^2}$$

$$\frac{\partial w}{\partial r} = -\frac{a_1 t\varepsilon}{\pi} \frac{\partial \theta_2}{\partial r} + \frac{\varepsilon}{\pi} \sqrt{R^2(\theta_2) + z_2} \frac{\partial \theta_2}{\partial r} + \frac{\varepsilon}{2\pi} \int_{-\theta_2}^{\theta_2} \frac{R \partial R / \partial r}{\sqrt{R^2 + z^2}}\, d\theta$$

Compute now $\partial w/\partial r$ for $z = 0$. In this case the integrals in the expression $\partial w/\partial r$ will simplify, and we obtain

$$J = \int_{-\theta_2}^{\theta_2} \frac{\partial R}{\partial r}\, d\theta = \frac{\partial}{\partial r} \int_{-\theta_2}^{\theta_1} R\, d\theta = \frac{\partial}{\partial r} \int_{-0}^{\theta_2} [r \cos\theta - \sqrt{r^2 \cos^2\theta + R_0{}^2 - r^2}]\, d\theta =$$

$$= 2 \sin\theta_2 - \frac{\partial}{\partial r} \int_{-\theta_2}^{\theta_2} \sqrt{r^2 \cos^2\theta + R_0{}^2 - r^2\, d\theta} = 2 \sin\theta_2 + r \int \frac{\sin^2\theta\, d\theta}{\sqrt{r^2 \cos^2\theta + R_0{}^2 - r^2}}$$

Translated by R. M. E.-I.